Journal of Engineering Mathematics 22: 73–92 (1988)

Similarity solutions in axisymmetric mixed-convection boundary-layer flow

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Received 25 June 1987; accepted 22 September 1987

Abstract. The similarity equations for mixed-convection axisymmetric boundary-layer flow are considered. The equations involve a buoyancy parameter α and a curvature parameter β. The equations are solved numerically and it is found that for large α, and β of O(1), an asymptotic solution is approached, the nature of which is discussed. When β is also large, of $O(\alpha^{1/4})$, the problem, at leading order, becomes independent of the mainstream and the free-convection limit is obtained. This problem is also discussed, including the behaviour for large values of β_0, the free-convection curvature parameter. For $\alpha < 0$ we find that the solution can be continued past the point where the wall heat transfer becomes zero (where previous mixed-convection similarity solutions in plane geometry were terminated) with the solution ending as $\alpha \to 0^-$. The nature of this limit is also discussed. For $\alpha < 0$ it is also found that there are solutions only in $\alpha_b \leqslant \alpha < 0$ with two branches of solution bifurcating out of $\alpha = \alpha_b$, and values of α_b are computed for a range of β. The behaviour of the solution for large values of the curvature parameter β, and α of O(1), is discussed where it is shown that the solution proceeds in inverse powers of $\log \beta$.

1. Introduction

The problem of similarity solutions in mixed-convection boundary-layer flow in two-dimensional geometry was treated first by Cohen and Reshotko [1]. This work was later extended by Wilks and Bramley [2] who showed the existence of dual solutions, the bifurcation point of which was found to be distinct from the point of vanishing skin friction. They also examined numerically the eigenvalue problem arising out of a stability analysis of these solutions. Mixed-convection boundary-layer flows in axisymmetric geometries have received little attention so far. The problem of mixed-convection boundary-layer flow of a uniform stream past an isothermal vertical circular cylinder has been discussed by Mahmood and Merkin [3] for both cases when the buoyancy forces aid and oppose the development of the boundary layer. For axial incompressible boundary-layer flow on a circular cylinder, Stewartson [4] showed that a similarity solution was possible if the main stream $U(x)$ was of the form $U = U_0 x/l$, though he did not go on to discuss the solution of the resulting equation.

In this paper we consider the possible form of mainstream and cylinder temperature for which, in a fluid at constant ambient temperature T_0, the axisymmetric boundary-layer equations can be reduced to similarity form. This again requires that we take a main stream of the form $U(x) = U_0 x/l$ and that the temperature on the cylinder T_w have the form $T_w = T_0 + (x/l)\Delta T$. These similarity equations involve both the buoyancy parameter $\alpha = g\beta l \Delta T/U_0^2$ and the curvature parameter $\beta = (\nu l U_0^{-1} a^{-2})^{1/2}$, where a is the radius of the cylinder. With $\beta = 0$, the equations reduce to one of a class of equations discussed in [1, 2], and with $\alpha = 0$ the momentum equation becomes the equation given in [3].

We begin by considering the case $\alpha > 0$ (aiding flow) and solve the resulting equations numerically for a range of values of β. We then go on to obtain an asymptotic solution for $\alpha \gg 1$ and β of O(1). From this solution it then appears that for the curvature effects to be important at leading order when α is large, β must be of $O(\alpha^{1/4})$. We then take β to be of the form $\beta = \beta_0 \alpha^{1/4}$ and let $\alpha \to \infty$. With this choice of scaling for β, the problem becomes independent of the free stream depending only on the applied temperature difference, and so the resulting equations at leading order can be regarded as the free-convection limit of the mixed-convection problem. We solve this problem numerically for a range of β_0 and examine the behaviour of the solution for β_0 large.

For the opposing case, $\alpha < 0$, we find that, as with the case $\beta = 0$, [2], there is only a finite range of α, $\alpha_b \leqslant \alpha < 0$ (say) for which a solution is possible. Two branches of solutions bifurcate out of $\alpha = \alpha_b$ (the value of α_b depends on the value of the parameter β) with the upper branch continuing the solution into $\alpha > 0$. However, we find that the lower branch of solutions can be continued past the point of zero heat transfer from the cylinder, which was where the solutions given by [2] ended, and that this lower branch terminates as $\alpha \to 0^-$. We then obtained the asymptotic behaviour of this lower branch as $\alpha \to 0^-$. The effect of increasing β is then found to reduce the value of α at which the skin friction vanishes though the basic overall structure of the solutions remains the same; there still being two branches of solution and with the lower branch terminating as $\alpha \to 0^-$.

Finally we consider the case of large curvature, with α of O(1). Here we find the solution develops a logarithmic singularity at the wall, with the solution proceeding in inverse powers of $\log \beta$, and has some similarities with the asymptotic solution for the flow of a uniform stream over a circular cylinder as discussed by Stewartson [4] and Glauert and Lighthill [6].

2. Equations

The equations governing axisymmetric mixed-convection boundary-layer flow with main-stream $U(x)$ in fluid at constant ambient temperature T_0 are

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial r}(rw) = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial r} = U\frac{\mathrm{d}U}{\mathrm{d}x} + g\beta(T - T_0) + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \tag{2}$$

$$u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial r} = \frac{\nu}{\mathrm{Pr}}\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right), \tag{3}$$

where the coordinates x and r measure distance along the surface and normal to it respectively. The boundary conditions to be applied are

$$u = w = 0, \quad T = T_0 + \frac{\Delta T}{l}x \quad \text{on} \quad r = a;$$

$$u \to \frac{U_0 x}{l}, \quad T \to T_0 \quad \text{as} \quad r \to \infty; \tag{4}$$

u, w are velocity components in the x and r directions respectively and T is the temperature of the fluid in the boundary layer, β is the coefficient of thermal expansion, g is acceleration due to gravity, ν is the kinematic viscosity and Pr the Prandtl number.

With this choice of mainstream and cylinder temperature, equations (1)–(3) can be reduced to similarity form by putting

$$\psi = \nu x\left(\frac{U_0 a^2}{\nu l}\right)^{1/2} f(\eta), \quad T - T_0 = \frac{\Delta T}{l} x\theta(\eta),$$
$$\eta = \frac{r^2 - a^2}{2\nu l}\left(\frac{\nu l}{U_0 a^2}\right)^{1/2}, \tag{5}$$

where ψ is the stream function defined so that $u = r^{-1}\partial\psi/\partial r$ and $w = -r^{-1}\partial\psi/\partial x$. Equation (1) is then automatically satisfied and equations (2) and (3) become

$$(1 + 2\beta\eta)f''' + 2\beta f'' + ff'' + 1 - f'^2 + \alpha\theta = 0, \tag{6a}$$

$$(1 + 2\beta\eta)\theta'' + 2\beta\theta' + \Pr(f\theta' - f'\theta) = 0, \tag{6b}$$

together with the boundary conditions

$$f(0) = f'(0) = 0, \quad \theta(0) = 1;$$
$$f' \to 1, \quad \theta \to 0 \quad \text{as} \quad \eta \to \infty \tag{7}$$

(primes denote differentiation with respect to η).

Here $\alpha = g\beta l\Delta T/U_0^2$ is the buoyancy parameter (the ratio of free to forced convection velocity scales) and $\beta = (\nu l U_0^{-1} a^{-2})^{1/2}$ is the curvature parameter.

When the curvature parameter $\beta = 0$, equations (6) reduce to

$$f''' + ff'' + 1 - f'^2 + \alpha\theta = 0,$$
$$\Pr^{-1}\theta'' + f\theta' - f'\theta = 0, \tag{8}$$

together with boundary conditions (7). Equations (8) are one of a class of two-dimensional mixed-convection similarity equations discussed by Cohen and Reshotko [1] and Wilks and Bramley [2]. With $\alpha = 0$, equation (6a) was given by Stewartson [3] though its solution was not discussed in any detail.

The purpose of this paper is to discuss the solution of equations (6) together with boundary conditions (7) over the range of values of the parameters α and β. Throughout results will be given for $\Pr = 1$, though the methods presented will apply generally. We begin by considering the case $\alpha > 0$ (aiding flow).

3. Aiding case, $\alpha > 0$

Equations (6) were solved numerically for a range of $\alpha > 0$. Graphs of $f''(0)$ and $\theta'(0)$ are shown in Figs. (1a) and (1b) respectively for $\beta = 1$ and $\beta = 5$ and compared with the values for $\beta = 0$. We can see from these figures that the curves for $\beta = 1$ and $\beta = 5$ have the same general shape as that for $\beta = 0$, with the effect of increasing the curvature being, for a given value of α, to increase the skin friction $f''(0)$ and the heat transfer $-\theta'(0)$ on the cylinder.

To discuss the behaviour of the solution for $\alpha \gg 1$, we put

$$f = \alpha^{1/4}F, \quad \zeta = \alpha^{1/4}\eta. \tag{9}$$

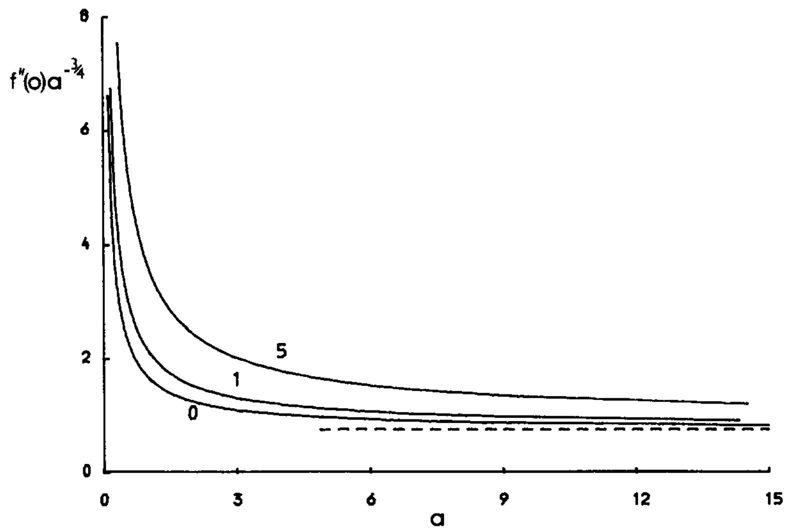

Fig. 1a. Graphs of $f''(0)\ \alpha^{-3/4}$ for $\beta = 0$, 1 and 5 (the asymptotic value is shown by the broken line).

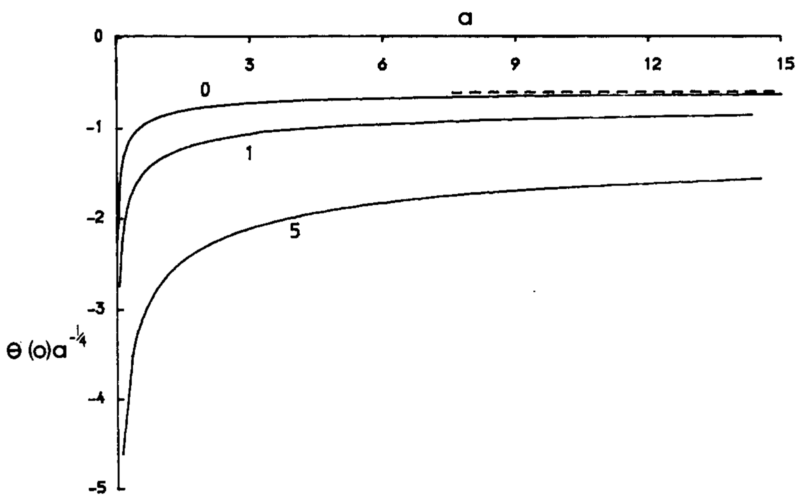

Fig. 1b. Graphs of $\theta'(0)\ \alpha^{-1/4}$ for $\beta = 0$, 1 and 5 (the asymptotic value is shown by the broken line).

Then with $F = F(\zeta)$ and $\theta = \theta(\zeta)$, equations (6) become

$$F''' + FF'' - F'^2 + \theta + 2\beta\alpha^{-1/4}(\zeta F''' + F'') + \alpha^{-1} = 0, \tag{10a}$$

$$\theta'' + F\theta' - F'\theta + 2\beta\alpha^{-1/4}(\zeta\theta'' + \theta') = 0 \tag{10b}$$

(primes now denote differentiation with respect to ζ).

The boundary conditions on $\zeta = 0$ are given by (7), with the outer boundary conditions becoming

$$F' \to \alpha^{-1/2}, \quad \theta \to 0 \quad \text{as} \quad \zeta \to \infty. \tag{11}$$

Equations (10) show that, if β is of O(1), the curvature effects are small, being of $O(\alpha^{-1/4})$ and (11) shows that forced-convection effects are also small, being of $O(\alpha^{-1/2})$ so that, at leading order we obtain the free-convection flat-plate solution. Equations (10) and (11) suggest an expansion for F and θ in the form

$$\begin{aligned} F &= F_0 + \alpha^{-1/4}\beta F_1 + \alpha^{-1/2}(\beta^2 F_2 + \phi_2) + \ldots, \\ \theta &= \theta_0 + \alpha^{-1/4}\beta\theta_1 + \alpha^{-1/2}(\beta^2\theta_2 + h_2) + \ldots. \end{aligned} \tag{12}$$

F_0 and θ_0 satisfy the equations

$$\begin{aligned} F_0''' + F_0F_0'' - F_0'^2 + \theta_0 &= 0, \\ \theta_0'' + F_0\theta_0' - F_0'\theta_0 &= 0, \end{aligned} \tag{13a}$$

with boundary conditions

$$\begin{aligned} &F_0(0) = F_0'(0) = 0, \quad \theta_0(0) = 1; \\ &F_0' \to 0, \quad \theta_0 \to 0 \quad \text{as} \quad \zeta \to \infty. \end{aligned} \tag{13b}$$

The equations for the higher-order terms in expansions (12) are all linear, with, at $O(\alpha^{-1/2})$, the β^2F_2 and $\beta^2\theta_2$ terms being associated with the curvature effects, and the terms ϕ_2 and h_2 being associated with the forced-convection effects. We find, from solving the resulting equations numerically that for $\alpha \gg 1$,

$$\begin{aligned} \left(\frac{d^2f}{d\eta^2}\right)_0 &= \alpha^{3/4}[0.739\,50 + 0.134\,62\beta\alpha^{-1/4} + (0.019\,89 - 0.047\,33\beta^2)\alpha^{-1/2} + \ldots], \\ \left(\frac{d\theta}{d\eta}\right)_0 &= -\alpha^{1/4}[-0.595\,09 + 0.470\,07\beta\alpha^{-1/4} + (0.042\,15 - 0.158\,67\beta^2)\alpha^{-1/2} \ldots]. \end{aligned} \tag{14}$$

Equations (14) show that, as $\alpha \to \infty, f''(0)\alpha^{-3/4} \to 0.739\,50$ and $\theta'(0)\alpha^{-1/4} \to -0.595\,09$. These asymptotic values are also shown in Figs. 1a and 1b (by the broken lines) and we can see that in each case the solution approaches this asymptotic value as $\alpha \to \infty$, though the rate of approach becomes slower the larger the value of β. This is to be expected, as we have from above, that if β is $O(\alpha^{1/4})$, curvature effects become important at leading order, and the nature of the solution changes. This is the case we discuss next.

4. The free-convection limit

We have seen above that for the curvature effects to be important at leading order when $\alpha \gg 1$, β must be of $O(\alpha^{1/4})$. So if we put $\beta = \beta_0\alpha^{1/4}$ and then let $\alpha \to \infty$, equations (10) become

$$F'''(1 + 2\beta_0\zeta) + 2\beta_0 F'' + FF'' - F'^2 + \theta = 0, \tag{15a}$$

$$\theta''(1 + 2\beta_0\zeta) + 2\beta_0\theta' + F\theta' - F'\theta = 0, \tag{15b}$$

with boundary conditions (11) becoming

$$\begin{aligned} &F(0) = F'(0) = 0 \quad \theta(0) = 1; \\ &F' \to 0, \quad \theta \to 0 \quad \text{as} \quad \zeta \to \infty. \end{aligned} \tag{16}$$

With this choice of scaling for β, the non-dimensionalisation given by (5) becomes independent of the free-stream, and depends only on the applied temperature difference. Hence we can regard equations (15) and (16) as the free-convection limit of the mixed-convection problem defined by equations (6) and (7). Equations governing the free-convection similarity solution on a vertical circular cylinder have previously been given by Millsaps and Pohlhausen, [10]. The equations derived in [10] are somewhat different to equations (15), (they used a different transformation of variables) and no detailed discussion of their solution was given.

Equations (15) were solved numerically for increasing value of β_0 (with $\beta_0 = 0$, we obtain equations (13) for which we have already obtained a solution). Graphs of $(d^2f/d\eta^2)_0 = \alpha^{3/4}F''(0)$ and $(d\theta/d\eta)_0 = \alpha^{1/4}\theta'(0)$ plotted against β_0 are shown in Figs. (2a) and (2b) respectively. From these figures we can see that both $F''(0)$ and $-\theta'(0)$ increase as β_0 is increased. The solution develops a pronounced double structure for the larger values of β_0. This can be seen from Figs. (2c) and (2d) where graphs of $F'(\zeta)$ and $\theta(\zeta)$ for $\beta_0 = 1, 5, 10$ are shown. For $\beta_0 = 5$ and more obviously for $\beta_0 = 10$, both F' and θ have a steep gradient near the wall (ζ small) followed by a long "tail" region before the outer boundary conditions are finally reached.

We now go on to consider the solution for β_0 large. To do this we first put $Y = \zeta + 1/(2\beta_0)$ with equations (15) becoming

$$2\beta_0 yF''' + 2\beta_0 F'' + FF'' - F'^2 + \theta = 0, \tag{17a}$$

$$2\beta_0 y\theta'' + 2\beta_0\theta' + F\theta' - F'\theta = 0, \tag{17b}$$

with boundary conditions

$$
\begin{aligned}
&F = F' = 0, \quad \theta = 1 \quad \text{on} \quad y = \frac{1}{2\beta_0}; \\
&\quad F' \to 0, \quad \theta \to 0 \quad \text{as} \quad y \to \infty
\end{aligned}
\tag{18}
$$

(primes now denote differentiation with respect to y).

We look for a transformation of equations (17) which will be valid when $\beta_0 \gg 1$. To do this we put $F = a\phi$, $\theta = bh$ and $Y = cy$, where a, b and c are functions of β_0 to be determined. A balancing of terms in the momentum equation (17a) then gives $a = \beta_0$ and $b = \beta_0^2 c^2$. Also equation (17b) becomes, for small Y, $Yh'' + h' \simeq 0$ so that $h \simeq A_0 \log Y + B_0$, (for constants A_0 and B_0) which in turn implies that $\theta \simeq b[A_0 \log Y + B_0]$ for $Y \ll 1$. The boundary condition on $y = 1/(2\beta_0)$, i.e. $Y = c/(2\beta_0)$, then requires $1 = b[A_0 \log \{c/(2\beta_0)\} + B_0]$. Hence we need to choose b so that $b = -1/\log(c/2\beta_0)$ (taking the negative sign, since $c/\beta_0 \ll 1$); c is then determined from $c^2 = [\beta_0^2 \log (2\beta_0/c]^{-1}$.

All this suggests the transformation

$$
F = \beta_0 \phi(Y), \quad \theta = \frac{h(Y)}{\log(1/\delta)} \quad \text{and} \quad Y = 2\beta_0 \delta y, \tag{19}
$$

with equation (17) becoming

$$
2Y\phi''' + 2\phi'' + \phi\phi'' - \phi'^2 + h = 0, \tag{20a}
$$

$$
2Yh'' + 2h' + \phi h' - \phi' h = 0, \tag{20b}
$$

Subject to the boundary conditions

$$
\begin{aligned}
&\phi = \phi' = 0, \quad h = \log(1/\delta) \quad \text{on} \quad Y = \delta; \\
&\quad \phi' \to 0, \quad h \to 0 \quad \text{as} \quad Y \to \infty
\end{aligned}
\tag{21}
$$

(primes now denote differentiation with respect to Y).

δ is related to β_0 via

$$
\delta^2 \log(1/\delta) = 1/4\beta_0^4, \tag{22}
$$

so that $\delta = (2\sqrt{2}\beta_0^2)^{-1} (\log \beta_0)^{-1/2} (1 + \ldots$. The transformation (19) is similar in many respects to that used by Kuiken [7] to describe the axisymmetric free-convection boundary-layer flow at large distances along an isothermal circular cylinder. As noted above for $Y \ll 1$, the solution for h is $h = A_0 \log Y + B_0 + \ldots$. So to satisfy the inner boundary condition on h, we must take $A_0 = -1$. Then this boundary condition is satisfied to $O(1/\log(1/\delta)$, which in turn means we require an expansion for ϕ and h in

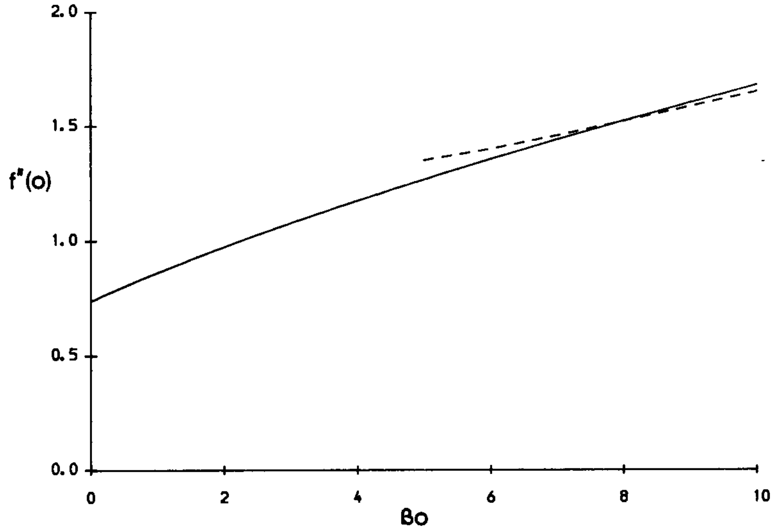

Fig. 2a. Graphs of $f''(0)$ against β_0 for the free-convection limit (asymptotic series (30b) is shown by the broken line).

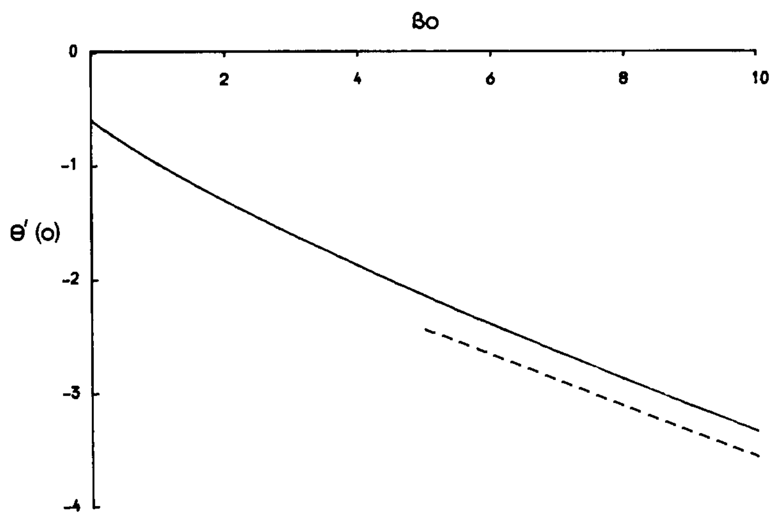

Fig. 2b. Graphs of $\theta'(0)$ against β_0 for the free-convection limit (asymptotic series (30a) is shown by the broken line).

the form

$$\begin{aligned} \phi &= \phi_0 + \frac{1}{\log(1/\delta)}\phi_1 + \frac{1}{(\log(1/\delta))^2}\phi_2 + \dots, \\ h &= h_0 + \frac{1}{\log(1/\delta)}h_1 + \frac{1}{(\log(1/\delta))^2}h_2 + \dots. \end{aligned} \tag{23}$$

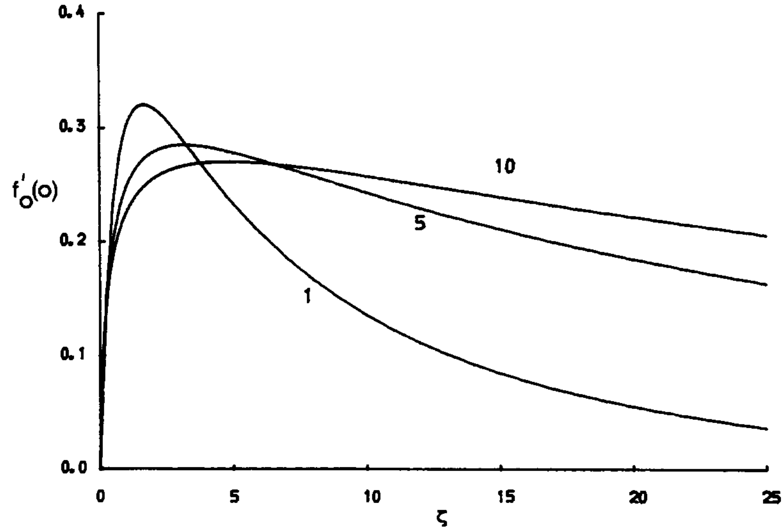

Fig. 2c. Velocity profiles f_0'' for $\beta_0 = 1$, 5 and 10 for the free-convection limit.

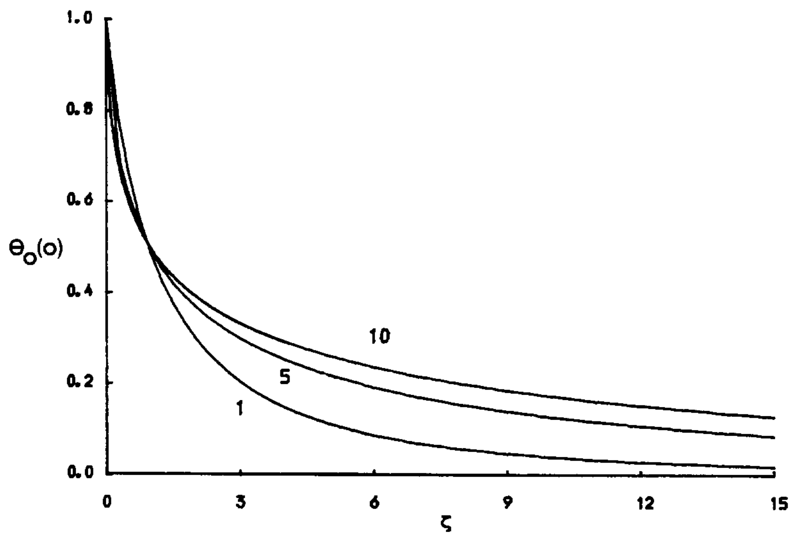

Fig. 2d. Temperature profiles θ_0 for $\beta_0 = 1$, 5 and 10 for the free-convection limit.

The equations at leading order are

$$\begin{aligned} 2Y\phi_0''' + 2\phi_0'' + \phi_0\phi_0'' - \phi_0'^2 + h_0 &= 0, \\ 2Yh_0'' + 2h_0' + h_0'\phi_0 - \phi_0'h_0 &= 0, \end{aligned} \tag{24}$$

with boundary conditions

$$h_0 \to 0, \quad \phi_0 \to 0 \quad \text{as} \quad Y \to \infty, \tag{25}$$

and for small Y

$$h_0 \sim -\log Y + B_0 + \ldots . \tag{26}$$

For ϕ_0 we then have, approximately, for $Y \ll 1$, $2Y\phi_0''' + 2\phi_0'' \simeq \log Y - B_0$ so that $\phi_0' = C_0 + D_0 \log Y + O(Y \log Y)$. We need a solution which is not singular at $Y = 0$ so $D_0 = 0$ and

$$\phi_0 = C_0 Y + \ldots . \tag{27}$$

From (27) the inner boundary condition on ϕ is then satisfied to $O(\delta)$, which is correct to an order of magnitude much smaller than for the series expansion (23).

We then have to find a solution to equations (24) which satisfies (26) and (27) for $Y \ll 1$ and (25) as $Y \to \infty$. This has to be done numerically, and because of the singularity at $Y = 0$, we have to start the integration at a small but non-zero value of Y, using expansions for ϕ_0 and h_0, valid for small Y, namely

$$\begin{aligned} \phi_0 &= C_0 Y + \tfrac{1}{4} Y^2 \log Y + \tfrac{1}{8}(2C_0^2 - 2B_0 - 5)Y^2 + \ldots, \\ h_0 &= -\log Y + B_0 - \tfrac{1}{2} C_0 Y \log Y + \tfrac{1}{2}(B_0 C_0 + 3C_0)\, Y + \ldots . \end{aligned} \tag{28}$$

Using (28), starting the integration at $Y = 0.01$, and applying the outer boundary condition at $Y = 150$, we found that $B_0 = -0.6602$, $C_0 = 0.9992$. Other initial and final values for Y were tried, and these we found to give the same values for B_0 and C_0 to the accuracy quoted. Now at $O(1/\log(1/\delta))$, we will have, for $Y \ll 1$,

$$h_1 \simeq A_1 \log Y + B_1, \quad \phi_1' \simeq D_1 \log Y + C_1, \tag{29}$$

and satisfying the boundary condition on $Y = \delta$ gives $A_1 = B_0$, $D_1 = C_0$; and the expansions can then be developed. On solving the linear equations for ϕ_1 and h_1 using the same method as for ϕ_0 and h_0 (i.e., by starting the solution off for small Y by a series expansion) we found that $C_1 = 1.7398$ and $B_1 = 0.7551$. Using (19), (26), (27) and (29), we find that, for $\beta_0 \gg 1$,

$$\left(\frac{d\theta}{d\zeta}\right)_{\zeta=0} = -\frac{2\beta_0}{\log(1/\delta)}\left(1 - \frac{B_0}{\log(1/\delta)} + \cdots\right), \tag{30a}$$

$$\left(\frac{d^2F}{d\zeta^2}\right)_{\zeta=0} = \frac{4\beta_0^3\delta}{\log(1/\delta)}\left(C_0 + \frac{C_1}{\log(1/\delta)} + \cdots\right). \tag{30b}$$

To use the asymptotic expressions (30) for $(d^2F/d\zeta^2)_{\zeta=0}$ and $(d\theta/d\zeta)_{\zeta=0}$ we first need to find δ in terms of β_0. This was done by using Newton's method to solve expressions (22) numerically for a range of values of β_0. These values of δ could then be used in (30) and graphs of $(d^2F/d\zeta^2)_{\zeta=0}$ and $(d\theta/d\zeta)_{\zeta=0}$ drawn using these asymptotic expressions. These are also shown in Figs. (2a) and (2b) respectively (by the broken lines). We can see that they are in good agreement with the values obtained by solving equations (15) numerically.

5. Opposing case, $\alpha < 0$

Consider first the case when $\beta = 0$, giving equations (8). The numerical solutions of these equations reveal that, as shown by [2], there is a value of α, α_b (say), which depends on the value of β, such that there are two branches of solution for $\alpha_b \leqslant \alpha < 0$ and no solution for $\alpha < \alpha_b$. However, we find that the lower branch of solutions can be continued past the point where $\theta'(0) = 0$, which was where the solutions given by [2] ended, and that this lower branch terminates as $\alpha \to 0^-$. Graphs of $f''(0)$ and $\theta'(0)$ for this case are shown in Figs. (3a) and (3b) respectively. From these figures we see that $f''(0)$ approaches a finite limit while $\theta'(0) \to \infty$ as $\alpha \to 0^-$.

Typical velocity and temperature profiles are shown in Figs. (3c) and (3d) respectively. These together with Fig. (3a) show that, for the lower branch of solutions, as $|\alpha|$ is decreased the region of reversed flow which initially appears next to the wall is moved away from the wall to a position within the boundary layer, with the velocity near the wall, and at large distances being positive. Graphs of $f''(0)$ and $\theta'(0)$ for $\beta = 1$ and $\beta = 5$ plotted against α are also shown in Figs. (3a) and (3b) respectively. From these figures we can see that the effect of increasing the curvature parameter β is to increase $f''(0)$ on the upper branch of solutions, and to decrease $f''(0)$ on the lower branch, with the value of $\theta'(0)$ being increased for a given value of α. By $\beta = 1$ the region of forward flow next to the wall has disappeared and there is now no second point of flow reversal.

To obtain a fuller understanding of how the curvature parameter affects the solution for $\alpha < 0$, we computed α_s, the value of α at which the skin friction vanished, i.e. at $\alpha = \alpha_s$, $f''(0) = 0$. To do this we solved equations (6) numerically with now the extra boundary condition that $f''(0) = 0$, and using the value of α as the undetermined parameter in the system. The value of this parameter and that of $\theta'(0)$ were then adjusted iteratively until the outer boundary conditions (7) were satisfied to sufficient accuracy. In this way, we could start at $\beta = 0$ where good estimates had already been obtained for α_s and $\theta'(0)$ and compute α_s (and $\theta'(0)$) for increasing values of β. A graph of α_s thus obtained is shown in Fig. (3e). This figure shows that $|\alpha_s|$ increases as β increases.

We also calculated the bifurcation point, α_b, of the dual solutions. To do this we put $\alpha = \alpha_b - \varepsilon$, where $0 < \varepsilon \ll 1$, and, as in [8, 9], we expand f and θ as

$$\begin{aligned} f &= f_0 + \varepsilon^{1/2} f_1 + \varepsilon f_2 + \dots, \\ \theta &= \theta_0 + \varepsilon^{1/2}\theta_1 + \varepsilon\theta_2 + \dots . \end{aligned} \tag{31}$$

f_0, θ_0 satisfy equations (6) and boundary conditions (7) with $\alpha = \alpha_b$. At $O(\varepsilon^{1/2})$, f_1, θ_1 satisfy the homogeneous problem

$$\begin{aligned} (1 + 2\beta\eta)f_1''' + 2\beta f_1'' + f_0 f_1'' - 2f_0' f_1' + f_0'' f_1 + \alpha_b\theta_1 &= 0, \\ (1 + 2\beta\eta)\theta_1'' + 2\beta\theta_1' + f_0\theta_1' + f_1\theta_0' - f_0'\theta_1 - f_1'\theta_0 &= 0, \end{aligned} \tag{32}$$

subject to the boundary conditions

$$f_1(0) = f_1'(0) = \theta_1(0) = f_1'(\infty) = \theta_1(\infty) = 0. \tag{33}$$

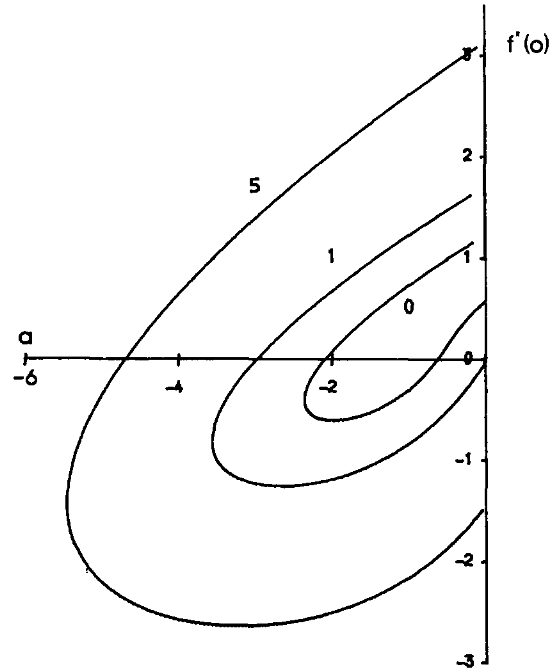

Fig. 3a. Graphs of $f''(0)$ for $\beta = 0$, 1 and 5 for the opposing case, $\alpha < 0$.

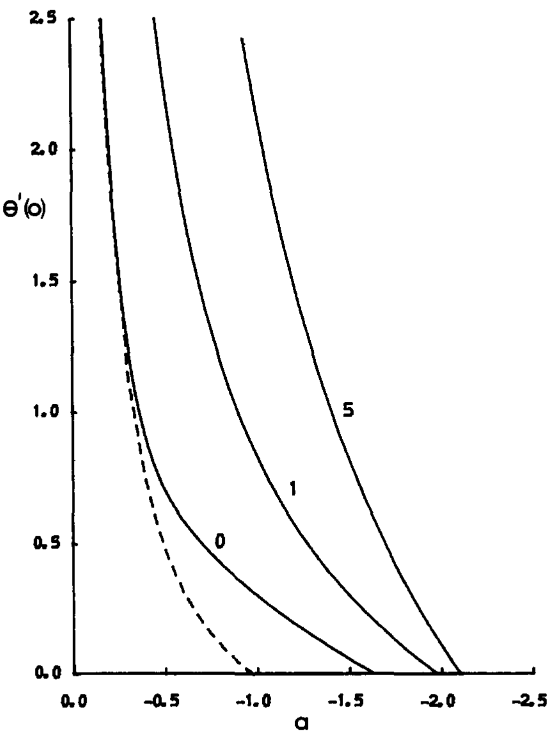

Fig. 3b. Graphs of $\theta'(0)$ for $\beta = 0$, 1 and 5 for the opposing case, $\alpha < 0$. The behaviour as $\alpha \to 0^-$ given by (46) is shown by the broken line.

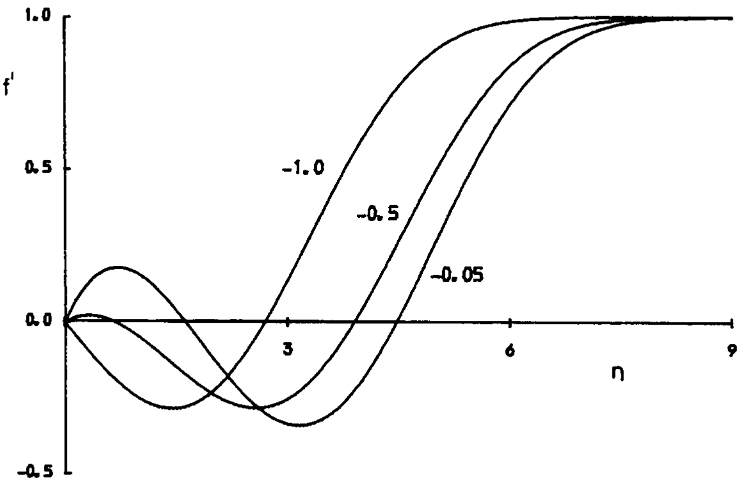

Fig. 3c. Velocity profiles f' for $\alpha = -1.0$, $\alpha = -0.5$ and $\alpha = -0.05$ with $\beta = 0$.

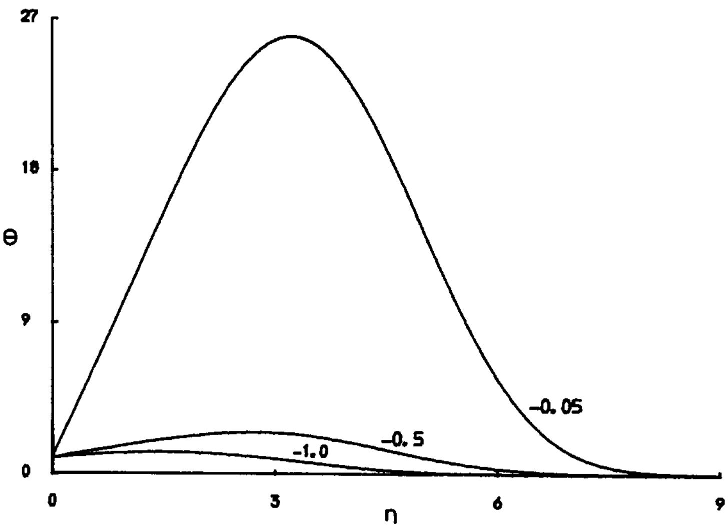

Fig. 3d. Temperature profiles θ for $\alpha = -1.0$, $\alpha = -0.5$ and $\alpha = -0.05$ with $\beta = 0$.

Equations (32) were solved by applying the extra boundary condition that $f_1''(0) = 1$ to force a non-trivial solution and using the value of α_b as an extra parameter, in a way similar to the determination of α_s described above. It was the solution of this homogeneous problem that determined α_b. The general solution of this problem will then be $f_1 = \kappa\bar{f}$, $\theta_1 = \kappa\bar{\theta}$ where $\bar{f}$, $\bar{\theta}$ is that solution which has $\bar{f}''(0) = 1$. The equations for the terms of $O(\varepsilon)$ are then

$$(1 + 2\beta\eta)f_2''' + 2\beta f_2'' + f_0 f_2'' - 2f_0' f_2' + f_0'' f_2 + \alpha_b\theta_2 = \kappa^2(\bar{f}'^2 - \bar{f}\bar{f}'') + \theta_0, \quad (34a)$$

$$(1 + 2\beta\eta)\theta_2'' + 2\beta\theta_2' + f_0\theta_2' + f_2\theta_0' - f_0'\theta_2 - f_2'\theta_0 = \kappa^2(\bar{f}'\bar{\theta} - \bar{f}\bar{\theta}'), \quad (34b)$$

with boundary conditions

$$f_2(0) = f_2'(0) = \theta_2(0) = f_2'(\infty) = \theta_2(\infty) = 0. \quad (35)$$

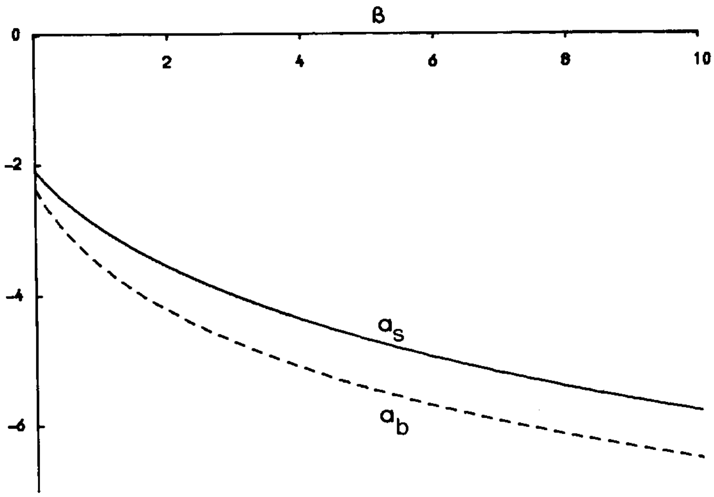

Fig. 3e. Graphs of α_s (the value of α where $f''(0) = 0$) and α_b (the value of α at the bifurcation of solutions) plotted against β.

To solve equations (34) we construct two particular integrals, one (f_a, θ_a) which has $f_a''(0) = \theta_a'(0) = 0$ and satisfies equations (34) with $\kappa = 1$ and the term θ_0 omitted from equation (34a), and a second, (f_b, θ_b) which has $f_b''(0) = \theta_b'(0) = 0$ with $\kappa = 0$. We also construct two complementary functions (f_c, θ_c) and (f_d, θ_d) which have $f_c''(0) = 1$, $\theta_c'(0) = 0$ and $f_d''(0) = 0$, $\theta_d'(0) = 1$. The full solution is then

$$\begin{aligned} f_2 &= \kappa^2 f_a + f_b + \lambda f_c + \mu f_d, \\ \theta_2 &= \kappa^2\theta_a + \theta_b + \lambda\theta_c + \mu\theta_d, \end{aligned} \tag{36}$$

for some constants λ and μ. Now, as $\eta \to \infty$, $f_i \sim B_i\eta^2$ and $\theta_i \sim A_i\eta$ $(i = a, b, c, d)$, so that to satisfy the outer boundary conditions we must choose λ and μ so that

$$\kappa^2 A_a + A_b + \lambda A_c + \mu A_d = 0, \tag{37a}$$

$$\kappa^2 B_a + B_b + \lambda B_c + \mu B_d = 0. \tag{37b}$$

But since there is a non-trivial solution to the equation at $O(\varepsilon^{1/2})$ there must be non-zero values of λ and μ such that $\lambda A_c + \mu A_d = 0$, $\lambda B_c + \mu B_d = 0$, i.e., $A_c B_d - A_d B_c = 0$. Using this, we can see that equations (37a) and (37b) are compatible only if

$$\kappa^2 = \frac{B_b A_c - A_b B_c}{A_a B_c - B_a A_c}. \tag{38}$$

Equation (38) has two solutions $\pm\kappa_0$ (say), and hence close to the bifurcation point α_b,

$$\begin{aligned} f''(0) &= f_0''(0) \pm \kappa_0(\alpha_b - \alpha)^{1/2} + \ldots, \\ \theta'(0) &= \theta_0'(0) \pm \kappa_0\bar{\theta}_1'(0)(\alpha_b - \alpha)^{1/2} + \ldots. \end{aligned} \tag{39}$$

(39) gives a square-root behaviour at the bifurcation point, with the + sign giving the start of the upper branch and the − sign the lower branch.

The values of α_b were determined (as described above) for a range of β, the results being shown in Fig. (3e). From this figure we can see that $|\alpha_b|$ is always greater than $|\alpha_s|$, showing that the general shape of the curves for $f''(0)$ will be essentially the same as those presented for $\beta = 0$, 1 and 5. Also for the larger values of β, the curves of α_s and α_b appear to the parallel.

Next consider the limit $\alpha \to 0^-$. The numerical solutions of equations (6) show that the form of f' changes only slightly as $\alpha \to 0^-$ whereas θ changes rapidly. So to obtain the form of the solution as $\alpha \to 0^-$, f and η are left unscaled and then to retain the buoyancy force term in equation (6a), we put $\theta = |\alpha|^{-1}\bar{h}(\eta)$, and look for a solution as $\alpha \to 0^-$. Equations (6) become

$$\begin{aligned}(1 + 2\beta\eta)f''' + 2\beta f'' + ff'' + 1 - f'^2 - \bar{h} &= 0,\\ (1 + 2\beta\eta)\bar{h}'' + 2\beta\bar{h}' + f\bar{h}' - f'\bar{h} &= 0,\end{aligned} \tag{40}$$

with boundary conditions (7) becoming

$$\begin{aligned}&f(0) = f'(0) = 0, \quad \bar{h}(0) = |\alpha|;\\ &f' \to 1, \quad \bar{h} \to 0 \quad \text{as} \quad \eta \to \infty\end{aligned} \tag{41}$$

(primes again denote differentiation with respect to η).
Boundary conditions (41) suggest an expansion of the form

$$\begin{aligned}f &= f_0 + |\alpha| f_1 + |\alpha|^2 f_2 + \dots,\\ \bar{h} &= \bar{h}_0 + |\alpha|\bar{h}_1 + |\alpha|^2\bar{h}_2 + \dots .\end{aligned} \tag{42}$$

The leading-order equations are

$$\begin{aligned}(1 + 2\beta\eta)f_0''' + 2\beta f_0'' + f_0 f_0'' + 1 - f_0'^2 - \bar{h}_0 &= 0,\\ (1 + 2\beta\eta)\theta_0'' + 2\beta\theta_0' + f_0\bar{h}_0' - f_0'\bar{h}_0 &= 0,\end{aligned} \tag{43}$$

with boundary conditions

$$\begin{aligned}&f_0(0) = f_0'(0) = \bar{h}_0(0) = 0;\\ &f_0' \to 1, \quad \bar{h}_0 \to 0 \quad \text{as} \quad \eta \to \infty.\end{aligned} \tag{44}$$

The equations for the higher-order terms in expansion (42) are all linear and can easily be solved once the solution of (43) is known. The solution with $\beta = 0$ gives

$$f''(0) = 0.57700 - 0.67360\,|\alpha| + \dots, \tag{45}$$

$$\theta'(0) = \frac{1}{|\alpha|}[0.49911 - 0.51130\,|\alpha| + \dots] \tag{46}$$

for $|\alpha| \ll 1$. Values of $\theta'(0)$ as calculated from (46) are also shown in Figure (3b) (by the broken line) and are seen to be in excellent agreement with the numerical solutions of equations (6).

6. Large curvature

In this section we look for a solution of equations (6) valid when the curvature parameter β is large with the buoyancy parameter α of O(1). To do this we first define a new independent variable $\bar{y}$ by $\bar{y} = 2\eta + \beta^{-1}$ and on putting $f = \frac{1}{2}\bar{f}$ equations (6) then become

$$\beta\bar{y}\bar{f}''' + \beta\bar{f}'' + \tfrac{1}{4}(\bar{f}\bar{f}'' + 1 - \bar{f}'^2) + \frac{\alpha}{4}\theta = 0, \tag{47a}$$

$$\beta\bar{y}\theta'' + \beta\theta' + \tfrac{1}{4}(\bar{f}\theta' - \bar{f}'\theta) = 0, \tag{47b}$$

with the boundary conditions

$$\bar{f} = \bar{f}' = 0, \quad \theta = 1 \quad \text{on} \quad \bar{y} = \beta^{-1};$$
$$\bar{f}' \to 1, \quad \theta \to 0 \quad \text{as} \quad \bar{y} \to \infty \tag{48}$$

(primes now denote differentiation with respect to $\bar{y}$).

We now look for a transformation of equations (47) which will hold when $\beta \gg 1$. Since equation (47b) shows that θ is of O(log $\bar{y}$) for $\bar{y}$ small, and then a balancing of the terms in equation (47a) together with the requirement that $\bar{f}' \to 1$ as $\bar{y} \to \infty$, leads us to the transformation

$$\bar{f} = \beta G(\tau), \quad \theta = \frac{1}{\log\beta}H(\tau), \quad \tau = \bar{y}/\beta. \tag{49}$$

Equations (47) become

$$\begin{aligned} &\tau G''' + G'' + \tfrac{1}{4}(GG'' + 1 - G'^2) + \frac{\alpha}{4\log\beta}H = 0, \\ &\tau H'' + H' + \tfrac{1}{4}(GH' - G'H) = 0 \end{aligned} \tag{50}$$

(primes denote differentiation with respect to τ).

With boundary conditions (48) now to be applied on $\tau = 1/\beta^2$. We look for a solution by expanding G and H in the form

$$G = G_0 + \frac{G_1}{\log\beta} + \frac{G_2}{(\log\beta)^2} + \ldots,$$

$$H = H_0 + \frac{H_1}{\log\beta} + \frac{H_2}{(\log\beta)^2} + \ldots. \tag{51}$$

At leading order, we obtain the equations

$$\tau G_0''' + G_0'' + \tfrac{1}{4}(G_0 G_0'' + 1 - G_0'^2) = 0, \tag{52}$$

$$\tau H_0'' + H_0' + \tfrac{1}{4}(H_0' G_0 - H_0 G_0') = 0. \tag{53}$$

The solution of equation (52) which satisfies $G_0' \to 1$ as $\tau \to \infty$ and is not singular at $\tau = 0$ is just

$$G_0 = \tau. \tag{54}$$

Using this equation, (53) becomes

$$\tau H_0'' + \left(\frac{\tau}{4} + 1\right) H_0' - \tfrac{1}{4} H_0 = 0. \tag{55}$$

We need a solution which has $H_0 \to 0$ as $\tau \to \infty$. The solution of equation (55) can be expressed in terms of confluent hypergeometric functions, Slater [5], as

$$H_0 = A_0 \, \mathrm{e}^{-\tau/4} U(2; 1; \tau/4) \tag{56}$$

for some constant A_0. Using results given in [5] we have, for $\tau \ll 1$, that $H_0 \sim -A_0[\log \tau - 2 \log 2 + \gamma + 1 + \ldots]$ where $\gamma = 0.577216$ is Euler's constant. This gives

$$\theta = -\frac{A_0}{\log \beta}(-2 \log \beta + (\gamma + 1 - 2 \log 2) + \ldots) \tag{57}$$

on $\tau = 1/\beta^2$, so that we must choose $A_0 = \tfrac{1}{2}$ to satisfy $\theta = 1$ at leading order. Then

$$H_0 = \tfrac{1}{2} \, \mathrm{e}^{-\tau/2} U(2; 1; \tau/4). \tag{58}$$

The equation for G_1 is then

$$\tau G_1''' + (1 + \tau/4) G_1'' - \tfrac{1}{2} G_1' = -\frac{\alpha}{8} \mathrm{e}^{-\tau/4} U(2; 1; \tau/4), \tag{59}$$

and we need a solution of equation (59) which is not exponentially large at ∞. The solution of equation (59) can also be written in terms of confluent hypergeometric functions as $G_1' = \mathrm{e}^{-\tau/4}\{(\alpha/2)U(2; 1; \tau/4) + B_1 U(3; 1; \tau/4)\}$. Since, from [5], $U(3; 1; \tau/4) \simeq -\tfrac{1}{2}(\log \tau + (\gamma + \tfrac{3}{2} - 2 \log 2) + \ldots)$ for $\tau \ll 1$, it follows that on $\tau = 1/\beta^2$,

$$G' = 1 + (\alpha_1 + B_1) + \mathrm{O}(1/\log \beta),$$

and this gives the constant $B_1 = -(1 + \alpha)$ and hence

$$G_1' = \mathrm{e}^{-\tau/4}\left\{\frac{\alpha}{2} U(2; 1; \tau/4) - (1 + \alpha) U(3; 1; \tau/4)\right\}. \tag{60}$$

We are now in a position to estimate the skin friction $(d^2f/d\eta^2)_0$ and heat transfer $(d\theta/d\eta)_0$ for $\beta \gg 1$. From (60), we have

$$G_1' \sim \tfrac{1}{2}\log\tau + \tfrac{1}{2}(\gamma + \tfrac{3}{2} - 2\log 2) + \frac{\alpha}{4} + \ldots \quad \text{for} \quad \tau \ll 1.$$

Then since $G_2' \sim B_2 \log\tau + \ldots$ for some constant B_2, we have, on satisfying the inner boundary condition on $\tau = 1/\beta^2$ that

$$B_2 = \tfrac{1}{4}(\gamma + \tfrac{3}{2} - 2\log 2) + \frac{\alpha}{8}.$$

Now from (49), $(d^2f/d\eta^2)_{\eta=0} = (2/\beta)(d^2G/d\tau^2)_{\tau=1/\beta^2}$ and hence, we obtain

$$\left(\frac{d^2f}{d\eta^2}\right)_{\eta=0} = \frac{\beta}{\log\beta}\left\{1 + \frac{\tfrac{1}{2}(\gamma + \tfrac{3}{2} - 2\log 2) + \alpha/4}{\log\beta} + \cdots\right\} \tag{61}$$

for $\beta \gg 1$.

A similar argument applies for H_1 which, using (58) will be of the form $H_1 \sim -\tfrac{1}{4}(\gamma + 1 - 2\log 2)\log\tau + \ldots$ for $\tau \ll 1$, and hence

$$\left(\frac{d\theta}{d\eta}\right)_{\eta=0} = -\frac{\beta}{\log\beta}\left(1 + \frac{\tfrac{1}{2}(\gamma + 1 - 2\log 2)}{\log\beta} + \cdots\right) \tag{62}$$

for $\beta \gg 1$.

Graphs of $(d^2f/d\eta^2)_{\eta=0}$ and $(d\theta/d\eta)_{\eta=0}$ obtained by solving equations (6) for the case $\alpha = 0$ are shown plotted against β in Figs. (4a) and (4b) respectively. Also shown on these graphs (by the broken lines) are the values of $(d^2f/d\eta^2)_0$ and $(d\theta/d\eta)_0$ as calculated from the

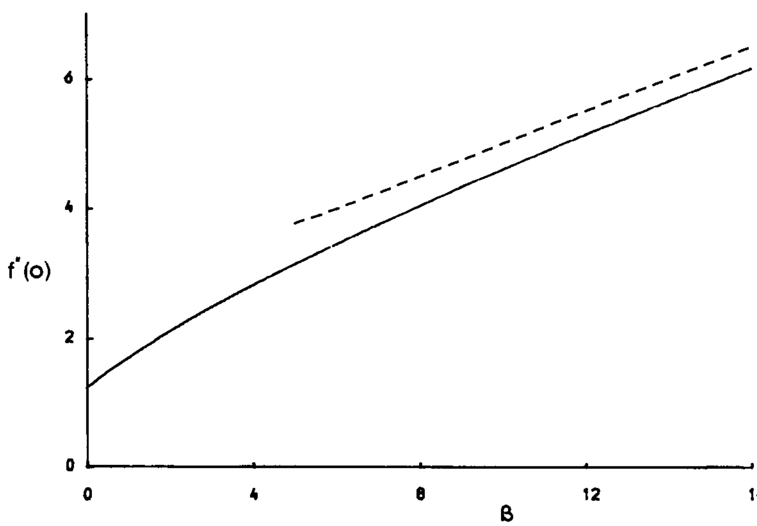

Fig. 4a. Graph of $f''(0)$ plotted against β with $\alpha = 0$. The large-curvature solution (61) is shown by the broken line.

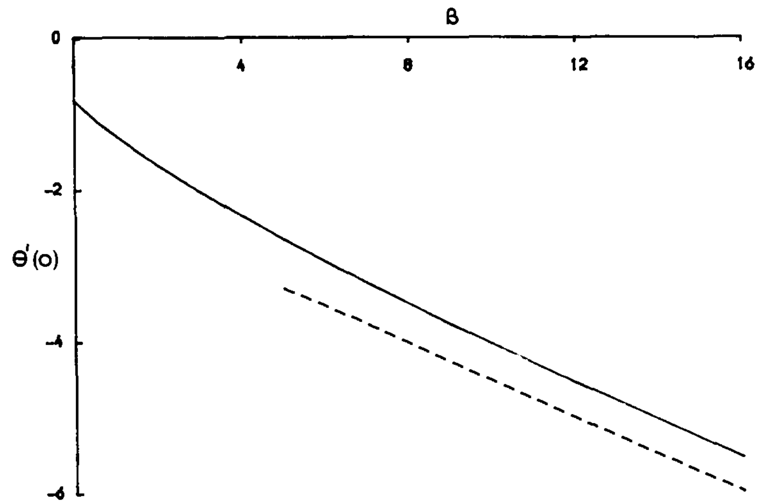

Fig. 4b. Graph of $\theta'(0)$ plotted against β with $\alpha = 0$. The large-curvature solution (62) is shown by the broken line.

asymptotic expressions (61) and (62) again with $\alpha = 0$. The figures clearly show the good agreement between these asymptotic expressions and the numerically determined values. Finally we notice that from (61), the value of α, α_s, where the skin friction goes to zero is of O(log β) for $\beta \gg 1$. (We cannot use (61) as it stands to estimate α_s more precisely since the next term in the series will involve an expession of the form $\alpha^2/(\log \beta)^2$ which will contribute to the estimate for α_s at leading order.) This is very slow increase in α_s for β large is in line with the numerically determined values shown in Fig. (3e).

Acknowledgement

One of us (T.M.) wishes to thank the Government of Pakistan for the grant to undertake this research.

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